

Zitterbewegung with spin-orbit coupled ultracold atoms in a fluctuating optical lattice

V. Yu. Argonov, D. V. Makarov

Pacific Oceanological Institute of the Russian Academy of Sciences,
43 Baltiiskaya st., 690041 Vladivostok, Russia, URL: <http://dynamab.poi.dvo.ru>

Abstract.

Dynamics of non-interacting ultracold atoms with artificial spin-orbit coupling is considered. Spin-orbit coupling is created using two moving optical lattices with orthogonal polarizations. Our main goal is to study influence of lattice noise on Rabi oscillations. Special attention is paid to the phenomenon of the Zitterbewegung being trembling motion caused by Rabi transitions between states with different velocities. Phase and amplitude fluctuations of lattices are modelled by means of the two-dimensional stochastic Ornstein-Uhlenbeck process, also known as harmonic noise. In the noiseless case the problem is solved analytically in terms of the momentum representation. It is shown that lattice noise significantly extends duration of the Zitterbewegung as compared to the noiseless case. This effect originates from noise-induced decoherence of Rabi oscillations.

1. Introduction

In recent years, enormous progress had been achieved in experimental realization of artificial gauge fields with ultracold atoms [1, 2]. It gives rise to the new era in quantum simulation of solid-state phenomena, such as the quantum Hall effect [3–5], spin-orbit coupling [6, 7], Majorana fermions [8], to name a few. This breakthrough is mainly conditioned by high degree of atom state controllability, allowing to reduce undesirable concomitant factors which lead to fast decoherence of quantum states in solid-state experiments. However, we have to account for mechanisms of decoherence which are relevant for ultracold atoms, like optical lattice amplitude noise [9, 10] or spontaneous emission [11–13].

In this work we consider dynamics of ultracold atoms with artificial spin-orbit coupling in the presence of optical lattice fluctuations. Attention is focused on influence of noise on Rabi inter-level oscillations. It is well-known that Rabi coupling for spin-orbit coupled states leads to onset of the Zitterbewegung (ZB) oscillations. The term ZB means specific trembling motion originally predicted by Schrödinger for relativistic Dirac electrons [14]. Basically, ZB occurs as a consequence of coupling between states with different velocities. The phenomenon of the ZB with ultracold atoms was considered, for example, in Refs. [15, 16]. We are interested in influence of lattice amplitude fluctuations on ZB oscillations. We consider the experimental setup where spin-orbit coupling is created by means of the Raman dressing scheme [6]. Lattice fluctuations are introduced using the harmonic noise [17, 18]. In the present work we restrict ourselves by the case of non-interacting atoms.

The paper is organized as follows. In Section 2 we introduce the model under consideration. Section 3 represents the analytical solution for Rabi oscillations in the absence of noise. Noise influence on Rabi oscillations is studied in Section 4. Effect of the Zitterbewegung is considered in Section 5. In Section 6 we summarize the results obtained and outline possible ways for further research.

2. Model

Consider gas of ultracold non-interacting ^{87}Rb atoms with mass M and momentum p moving in an external magnetic field and an optical field of two Raman lasers with orthogonal linear polarizations. Frequency difference between the two lasers is ω_L and the x -projection of wave vector difference is k_L . Magnetic field results in Zeeman splitting of the triplet state $F = 1$. Laser field induces transitions between three hyperfine states $m_F = -1$, $m_F = 0$, $m_F = 1$. Setting laser frequency difference ω_L to be close to

the frequency splitting ω_z between $m_F = -1$ and $m_F = 0$ states, we can neglect the level $m_F = 1$ due to the quadratic Zeeman effect. Therefore we have an effective two-level ($|m_F = -1\rangle$, $|m_F = 0\rangle$) system with the Hamiltonian [19]

$$\hat{H} = \frac{\hat{p}^2}{2M} + \frac{\hbar\Omega}{2}[\hat{\sigma}_1 U_1(x, t) + \hat{\sigma}_2 U_2(x, t)] - \frac{\hbar\omega_z}{2}\hat{\sigma}_3, \quad (1)$$

where $\hat{\sigma}_{1,2,3}$ are the Pauli matrices and Ω is the Rabi frequency (fixed by the intensity of the lasers). The Hamiltonian (1) is characterized by equal contributions of Rashba and Dresselhaus couplings. In the absence of lattice fluctuations, the terms U_1 and U_2 read

$$\begin{aligned} U_1(x, t) &= \cos(2k_L x - \omega_L t), \\ U_2(x, t) &= \sin(2k_L x - \omega_L t), \end{aligned} \quad (2)$$

Lattice fluctuations can be incorporated into the Hamiltonian by means of the replacement [20–22]

$$\begin{aligned} U_1(x, t) &= f(t) \cos(2k_L x) - sf(t + \tau) \sin(2k_L x), \\ U_2(x, t) &= f(t) \sin(2k_L x) + sf(t + \tau) \cos(2k_L x), \end{aligned} \quad (3)$$

where $f(t)$ is the two-dimensional Ornstein-Uhlenbeck process, also called harmonic noise. Harmonic noise is described by the coupled stochastic differential equations

$$\dot{f} = y, \quad \dot{y} = -\Gamma y - \omega_L^2 f + \sqrt{2\Gamma}\xi(t), \quad (4)$$

where Γ is a positive constant, and $\xi(t)$ is Gaussian white noise. It should be emphasized that terms $f(t)$ and $f(t + \tau)$ have to correspond to the same realization of noise ξ in order to provide wavelike form of U_1 and U_2 [20].

Power spectrum of harmonic noise has an unique peak at frequency

$$\omega_p = \sqrt{\omega_L^2 - \frac{\Gamma^2}{2}}. \quad (5)$$

Width of the peak is given by

$$\Delta\omega = \sqrt{\omega_p^2 + \Gamma\omega'} - \sqrt{\omega_p^2 - \Gamma\omega'}, \quad (6)$$

where

$$\omega' = \sqrt{\omega_L^2 - \frac{\Gamma^2}{4}}.$$

It is important to note that elimination of the hyperfine level $m_F = 1$ requires

$$|\omega_L - \omega_z| \gg \Delta\omega. \quad (7)$$

If $\Gamma \ll \omega_L$, then

$$\Delta\omega \approx \Gamma. \quad (8)$$

In the present work we consider values of Γ in the range $10^{-3}\omega_L \div 10^{-1}\omega_L$. If we set $\omega_L = 4.81$ MHz, as in [6], the corresponding laser bandwidth should be $10^3 \div 10^5$ Hz.

In the deterministic limit $\Gamma \rightarrow 0$ we have

$$f(t) \rightarrow \cos(\omega_L t + \phi_0).$$

Choosing initial conditions in (4) as $f(0) = 1$, $y(0) = 0$, we specify $\phi_0 = 0$. Then setting

$$\tau = \frac{\pi}{2\omega_L}, \quad (9)$$

one reproduces (2) if $\Gamma = 0$. Hence, it turns out that $U_1(x, t)$ and $U_2(x, t)$ for $\Gamma > 0$ behave as fluctuating plane waves. Let's denote

$$f(t) + if(t + \tau) = W(t)e^{-i\varphi(t)}. \quad (10)$$

In the deterministic case $\Gamma = 0$ we have $\varphi = \omega_L t$.

After the replacement (10), the Schrödinger equation can be represented as a pair of coupled equations

$$\begin{aligned} i\hbar \frac{\partial \Psi_a}{\partial t} &= \left(\frac{\hat{p}^2}{2M} - \frac{\hbar\omega_z}{2} \right) \Psi_a + \frac{\hbar\Omega W}{2} e^{i(\varphi - 2k_L x)} \Psi_b, \\ i\hbar \frac{\partial \Psi_b}{\partial t} &= \left(\frac{\hat{p}^2}{2M} + \frac{\hbar\omega_z}{2} \right) \Psi_b + \frac{\hbar\Omega W}{2} e^{-i(\varphi - 2k_L x)} \Psi_a, \end{aligned} \quad (11)$$

where Ψ_a and Ψ_b are wave functions of the $|m_F = -1\rangle$ and $|m_F = 0\rangle$ states, respectively.

Assume that dynamics is restricted by spatial domain of length L being integer multiple N of the lattice period, i. e.

$$L = \frac{\pi N}{k_L}, \quad N \gg 1. \quad (12)$$

Imposing periodic boundary conditions at the ends of the domain, we can define discrete set of momentum eigenfunctions

$$\psi_n = \frac{1}{\sqrt{L}} \exp(-ip_n x / \hbar), \quad (13)$$

where $p_n = n\hbar k_0$, n is integer, $k_0 = 2\pi/L$. Expanding Ψ_a and Ψ_b over momentum eigenfunctions

$$\Psi_a = \sum_n \alpha_n(t) \psi_n, \quad \Psi_b = \sum_n \beta_n(t) \psi_n, \quad (14)$$

we can significantly simplify the problem by reducing (11) to pairs of coupled ODE

$$\begin{aligned} i \frac{d\alpha_n}{dt} &= \frac{\Omega W}{2} \exp[i(\varphi - \nu_n t)] \beta_{n+l}, \\ i \frac{d\beta_{n+l}}{dt} &= \frac{\Omega W}{2} \exp[-i(\varphi - \nu_n t)] \alpha_n, \end{aligned} \quad (15)$$

where

$$\begin{aligned} \nu_n &= \frac{E_b(n+l) - E_a(n)}{\hbar}, \\ E_{a,b}(n) &= \frac{p_n^2}{2M} \mp \frac{\hbar\omega_z}{2}, \quad l = \frac{2k_L}{k_0}. \end{aligned} \quad (16)$$

For simplicity, we use scaling of variables corresponding to $M = 1$.

3. Noiseless case – analytical solution

In the purely deterministic case $\Gamma = 0$ solution of the equations (15) is given by

$$\begin{aligned} \alpha_n &= \sqrt{\rho_n} \left(c_{n1} e^{i\Omega_n^+ t} + c_{n2} e^{i\Omega_n^- t} \right) e^{i\gamma_n}, \\ \beta_{n+l} &= \sqrt{\rho_n} \left(c_{n3} e^{-i\Omega_n^- t} + c_{n4} e^{-i\Omega_n^+ t} \right) e^{i\gamma_n}, \end{aligned} \quad (17)$$

where γ_n is phase of a corresponding term in the expansion (14), c_{jn} are real-valued coefficients,

$$\Omega_n^\pm \equiv \frac{1}{2} \left(\chi_n \pm \tilde{\Omega}_n \right), \quad \tilde{\Omega}_n \equiv \sqrt{\chi_n^2 + \Omega^2}, \quad (18)$$

$$\chi_n \equiv \omega_L - \nu_n. \quad (19)$$

Constants c_{nj} obey the following relation:

$$c_{n1}^2 + c_{n2}^2 + c_{n3}^2 + c_{n4}^2 = 1. \quad (20)$$

Amplitudes α_n and β_{n+l} satisfy the conservation law

$$|\alpha_n|^2 + |\beta_{n+l}|^2 = \rho_n. \quad (21)$$

Substituting (17) into (15), we find the relations

$$c_{n3} = -\frac{2\Omega_n^+}{\Omega} c_{n1}, \quad c_{n4} = -\frac{2\Omega_n^-}{\Omega} c_{n2}. \quad (22)$$

We use the initial condition of a form

$$|\alpha_n(t=0)| = \sqrt{\rho_n}, \quad |\beta_n(t=0)| = 0 \quad (23)$$

for all n . Then we find

$$\begin{aligned} c_{n1} &= -\frac{\Omega_n^-}{\tilde{\Omega}_n}, \quad c_{n2} = \frac{\Omega_n^+}{\tilde{\Omega}_n}, \\ c_{n3} &= -\frac{\Omega}{2\tilde{\Omega}_n} = -c_{n4}. \end{aligned} \quad (24)$$

The case of

$$\chi_n = 0 \quad (25)$$

corresponds to onset of resonance in Eqs. (15), when $c_{n1} = c_{n2} = -c_{n3} = c_{n4} = 1/2$, therefore, α_n and β_{n+l} oscillate with frequency $\Omega/2$. Accordingly, we can regard the parameter χ_n as detuning of resonance between coupled momentum states. Increasing of χ_n results in growing of $|c_{n2}|$, while $|c_{n1}|$, $|c_{n3}|$ and $|c_{n4}|$ decrease.

Link between χ_n and corresponding momentum value can be easily derived from (16) and (19):

$$\chi_n = \omega_L - \omega_z - 2k_L p_n - 2\hbar k_L^2. \quad (26)$$

Any localized quantum state has finite width in the momentum space and therefore consists of many momentum states. In particular, we consider the initial state with Gaussian-like distribution

$$\rho_n = \frac{e^{-(p_n - p_c)^2 / 2\sigma_p^2}}{\sum_n e^{-(p_n - p_c)^2 / 2\sigma_p^2}}. \quad (27)$$

Owing to the linear dependence of χ_n on p_n , only one of the momentum states may satisfy the condition (25), other ones correspond to off-resonant transitions with nonzero values of χ_n .

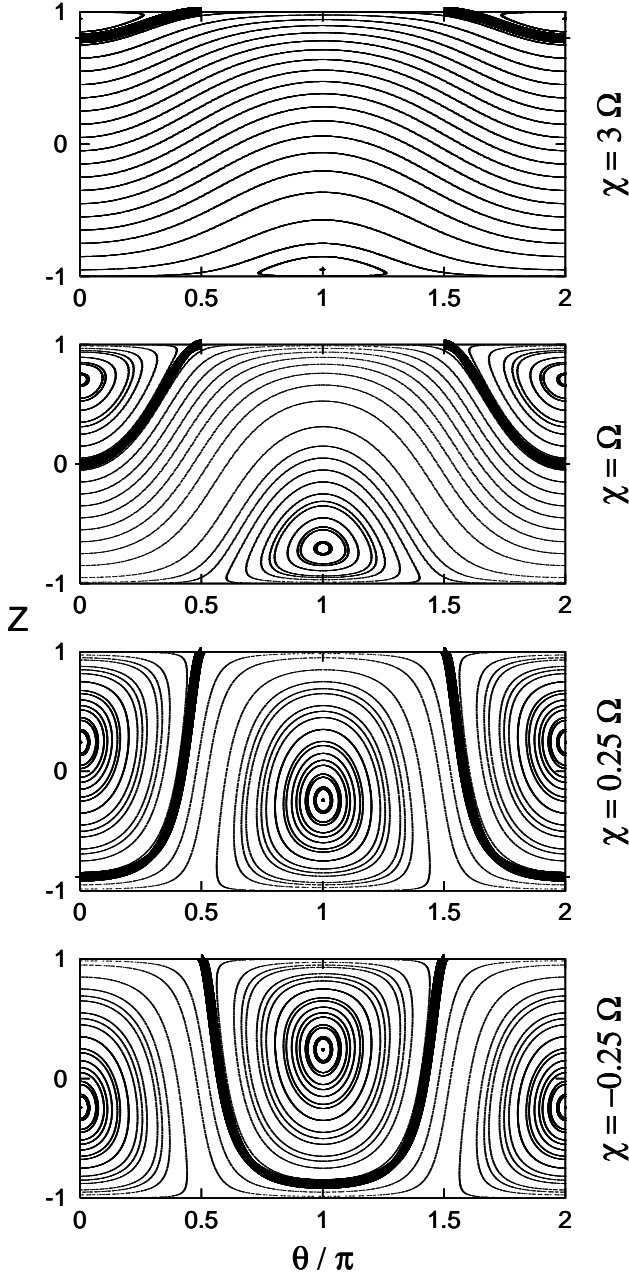


Figure 1. Phase portraits for the Hamiltonian (34) in the absence of noise. Values of the parameter χ are indicated to the right of each panel. Bold line corresponds to the solution (17) with coefficients determined by (24).

4. Phase space representation of Rabi oscillations

Rabi oscillations between resonantly coupled momentum components of $|m_F = -1\rangle$ and $|m_F = 0\rangle$ states can be readily described by means of normalized population imbalance for a n -th momentum state

$$Z(n, t) = \frac{|\alpha_n(t)|^2 - |\beta_{n+l}(t)|^2}{\rho_n} \quad (28)$$

After some simple algebra with the usage of (17) and (24), we find the solution for $Z(n, t)$ in the absence of noise

$$Z = \frac{1}{\Omega_n^2}(\chi_n^2 + \Omega^2 \cos \tilde{\Omega} t) \quad (29)$$

Further we omit the subscript n . From Eq. (29) it is clear that quantity $\tilde{\Omega}$ stands for effective Rabi frequency taking into account momentum-dependent detuning from the resonance (25).

In the presence of noise, dynamics of Z is governed by the equation that comes from (15) and looks as

$$\frac{dZ}{dt} = -\Omega W(t) \sqrt{1 - Z^2} \sin \theta, \quad (30)$$

where

$$\theta = \arg \alpha - \arg \beta - \varphi + \nu t. \quad (31)$$

Eq. (30) is complemented by equation

$$\frac{d\theta}{dt} = \Omega W(t) \frac{Z}{\sqrt{1 - Z^2}} \cos \theta - \chi - \eta_\chi(t). \quad (32)$$

where $\eta_\chi(t) = \dot{\varphi}(t) - \omega_L$ is a fluctuating part of the frequency difference between the Raman lasers. Equations (30) and (32) can be rewritten in the Hamiltonian form

$$\frac{dZ}{dt} = -\frac{\partial H}{\partial \theta}, \quad \frac{d\theta}{dt} = \frac{\partial H}{\partial Z}, \quad (33)$$

with the Hamiltonian

$$H = -\Omega(1 + \eta_w) \sqrt{1 - Z^2} \cos \theta - (\chi + \eta_\chi)Z, \quad (34)$$

where we used the replacement $W(t) = 1 + \eta_w(t)$. Form of the Hamiltonian (34) is generic for problems studying population dynamics. For example, a similar expression is used for describing population dynamics in external [23] and internal Bose-Josephson junctions [24, 25].

Stationary points of the Hamiltonian (34) in the absence of noise $\eta_w = \eta_\chi = 0$ can be found by solving the equations

$$\frac{dZ}{dt} = \frac{d\theta}{dt} = 0. \quad (35)$$

This gives

$$\theta_{st} = \pi(2k + 1), \quad Z_{st} = -\frac{\chi}{\Omega}, \quad (36)$$

and

$$\theta_{st} = 2\pi k, \quad Z_{st} = \frac{\chi}{\Omega}, \quad k \in \mathbb{Z}. \quad (37)$$

Eqs. (36) and (37) determine equilibrium points for Rabi oscillations with the dominance of the $|m_F = 0\rangle$ or $|m_F = -1\rangle$ states, depending on the sign of χ . Figure 1 illustrates phase portraits for Rabi oscillations with various values of χ . Equilibrium points are placed within the island-like phase space regions where phase θ is trapped. Despite the solution (17) goes by the trapping regions, the corresponding

trajectory lies predominantly in the upper half-plane, inferring prevalence of the state $|m_F = -1\rangle$. The only exception is the case of exact resonance $\chi = 0$ (not shown in the Figure). Dominance of the state $|m_F = -1\rangle$ is especially apparent in the case of $\chi = 3\Omega$, when the trajectory only slightly deviates from the upper bound $Z = 1$. Indeed, far from resonance (25) right-hand side terms in (15) rapidly oscillate with time and can be averaged out. Therefore, Rabi oscillations are suppressed. A similar phenomenon, referred to as the coherent population trapping, was earlier observed in [26–29] for two- and three-level atoms moving in a standing-wave laser field.

As noise is turned on, a phase space trajectory corresponding to Rabi oscillations is able to diffuse in phase space. So, it can drift to a region with different population dynamics, or even enter a trapping region. This leads to intermittency in Rabi oscillations. Phase space diffusion tends to mix the regimes where one of the state dominates, therefore, one may expect that time- and ensemble-averaged population imbalance

$$\bar{Z} = \frac{1}{N_r T} \sum_{j=1}^{N_r} \int_0^T \bar{Z}^{(j)}(t) dt, \quad (38)$$

should be closer to zero in the presence of noise than in the noiseless case. In Eq. (38), superscript j labels the realizations of harmonic noise, and N_r is total number of realizations. We calculate \bar{Z} for initial conditions (23). In the absence of noise, it can be easily found from (28) that

$$\bar{Z} = \frac{\chi^2}{\chi^2 + \Omega^2}. \quad (39)$$

The corresponding curve is shown in Fig. 2 by the bold line. As χ approaches to resonance, \bar{Z} decreases from 1 to 0 at $\chi = 0$. Thus, we can regard function $\bar{Z}(\chi)$ as resonance curve describing the process of the photon absorption by a two-level system. Numerical simulation in the presence of noise shows that intermittency reduces values of the time-averaged population imbalance as compared to the noiseless case. This leads to remarkable broadening of the resonance absorption peak, as Γ grows. This is especially pronounced in the case of $\Gamma = 0.1\Omega$.

5. Zitterbewegung

Population imbalance is related to a momentum expectation value $p(t) = \langle \Psi_a | \hat{p} | \Psi_a \rangle + \langle \Psi_b | \hat{p} | \Psi_b \rangle$ by means of the formula

$$p(t) = p(t=0) + \hbar k_L \left[1 - \sum_n \rho_n Z(n, t) \right], \quad (40)$$

where it is taken into account that $p_{n+1} - p_n = 2\hbar k_L$. As we use scaling corresponding to $M = 1$, p is equal to

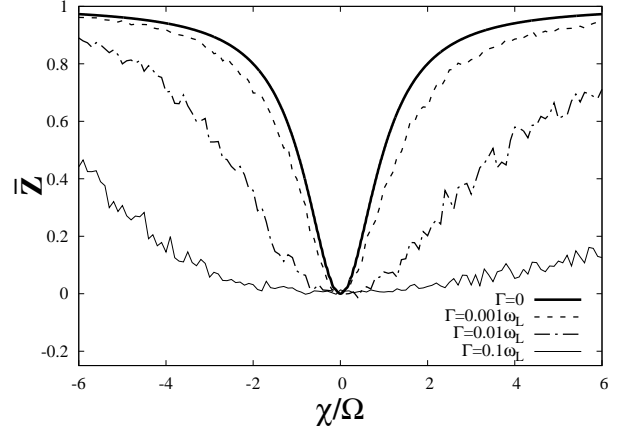


Figure 2. Time-averaged population imbalance between the $|m_F = -1\rangle$ and $|m_F = 0\rangle$ states for the n -th momentum state as function of the parameter χ_n defined by (19). Corresponding values of the harmonic noise parameter Γ are indicated in the corner of the Figure. Parameters values: $\omega_L = 10\Omega$, length of the interval for time averaging is $T = 100\pi/\Omega$. The curves for nonzero Γ correspond to averaging over 120 realizations of harmonic noise.

velocity of center-of-mass motion. As it follows from (40), Rabi inter-level transitions result in oscillations of p , hence, there occurs trembling motion referred to as the Zitterbewegung (or shortly ZB). Form of ZB oscillations depends on the initial momentum distribution (27). Firstly, it depends on the width of the distribution, as interference of terms with different n in (40) should damp ZB. In numerical simulation, we set $\sigma_p = 0.01\hbar k_L$. Owing to the Heisenberg uncertainty relation, it corresponds to $\sigma_x = \hbar/(2\sigma_p) \simeq 8\lambda_L$, where λ_L is laser wavelength. The center of the momentum distribution p_c is taken of 0. Secondly, oscillations of p depend on the detuning of the initial state from the resonance (25). Let's introduce mean detuning as

$$\chi_c = \omega_L - \omega_z - 2k_L p_c - 2\hbar k_L^2. \quad (41)$$

Figure 3 represents ZB oscillations in the absence of noise for $\chi_c = 0$ (upper panel) and $\chi_c = 0.5\Omega$ (lower panel). Notably, ZB exposes much faster damping in the case of $\chi_c = 0.5\Omega$ than in the case of $\chi_c = 0$ corresponding to the strongest influence of resonance.

Now let's consider effect of noise on ZB oscillations. In the presence of noise, strength of the ZB phenomenon can be quantified by amplitude of momentum oscillations. Therefore, it is reasonable to consider standard deviation of p within a temporal interval of length Δt

$$\Lambda_p(t) = \frac{1}{\hbar k_L} \sqrt{p^2 - \bar{p}^2}, \quad (42)$$

$$\bar{p}^k(t) \equiv \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} p^k(t') dt', \quad k = 1, 2.$$

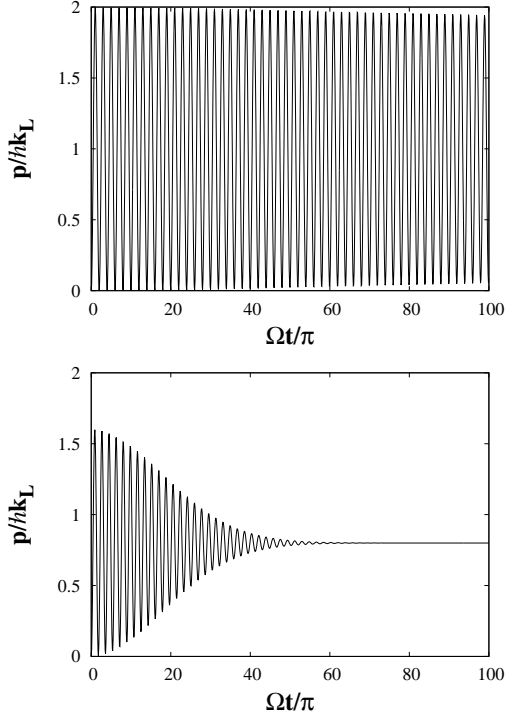


Figure 3. Quantum expectation value of momentum as function of time for $\chi_c = 0$ (upper panel) and $\chi_c = 0.5\Omega$ (lower panel).

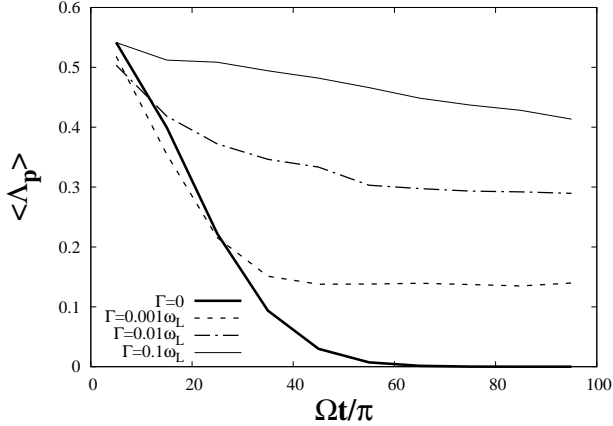


Figure 4. Standard deviation of momentum within a temporal interval of length $\Delta t = 10\pi/\Omega$. Values of other parameters: $\omega_L = 10\Omega$, $\chi_c = 0.5\Omega$.

The length of the temporal interval Δt has to be much larger than characteristic period of inter-level transitions $T \approx 2\pi/\Omega$. In numerical simulation we set $\Delta t = 10\pi/\Omega$. Data for Λ_p is averaged over 100 realizations of harmonic noise.

Evidently noise should add stochasticity into Rabi oscillations. One may expect that noise-induced phase fluctuations of Rabi oscillations can reduce phase coherence that is responsible for the ZB damping

in the noiseless case, in analogy with noise-induced destruction of the Anderson localization [21, 30, 31]. This expectation is fully confirmed by results of numerical simulation presented in Fig. 4. The case of $\chi_c = 0.5\Omega$ is considered. After decreasing within the initial stage, ensemble-averaged value of Λ_p achieves a plateau for all non-zero values of Γ , indicating persistence of ZB. Although increasing of Γ leads to higher values of Λ_p , effect of noise-induced ZB persistence is apparent even in the case of very weak noise $\Gamma = 10^{-3}\omega_L$. To understand the origin of the plateau, let's consider a typical realization of $p(t)$ for $\Gamma = 10^{-3}\omega_L$. Such an example is presented in Fig. 5. One can see that ZB oscillations firstly demonstrate damping, as in the noiseless case, but then there arises a burst with growing amplitude, as phase coherence of Rabi oscillations for different momentum eigenstates pairs is destroyed.

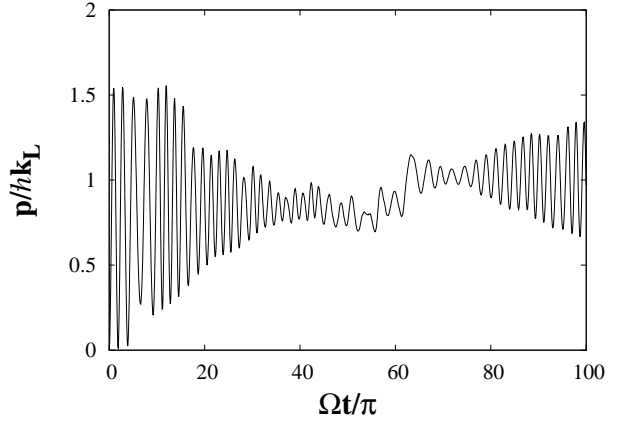


Figure 5. Momentum expectation value as function of time for a single realization of harmonic noise with $\Gamma = 0.001\omega_L$. The case of $\chi_c = 0.5\Omega$.

6. Summary

In the present work we consider dynamics of non-interacting ultracold atoms with artificial spin-orbit coupling imposed. Spin-orbit coupling is realized by means of the Raman dressing. Atoms move in a superposition of two moving optical lattices which experience random amplitude and phase fluctuations. Our main concern is to study influence of fluctuations on inter-level Rabi oscillations, with particular emphasis on the phenomenon of the Zitterbewegung (ZB). We find an analytical solution for the noiseless case. It is shown that dynamics of Rabi oscillations can be represented by an autonomous Hamiltonian system with one degree of freedom. As detuning from resonance (25) is significantly deviated from zero, there can occur the effect of the population

trapping. In the presence of noise the corresponding Hamiltonian system has $3/2$ degrees of freedom. Noise results in the onset of intermittency in Rabi oscillations. The main result of the paper is the noise-induced deceleration of ZB damping that is observed even if noise is weak. We link this effect with destruction of coherence in Rabi oscillations for different momentum components.

We model the fluctuations of a laser field by means of the harmonic noise. It implies that they are caused by some uncontrollable factors reducing laser coherence. However, the same role can be played by properly chosen broadband modulation of the laser wave. We expect that the effect of noise-induced persistence for ZB oscillations can be observed with broadband deterministic signals as well. Also, it is very interesting how this effect will change in the presence of interaction between atoms, for example, within a mean-field picture. We intend to address these issues in the forthcoming works.

Acknowledgments

This work was partially supported by the Russian Foundation of Basic Research within the projects 15-02-08774-a and 15-02-00463-a.

References

- [1] Lin Y J, Compton R L, Jimenez-Garcia K, Porto J V and Spielman I B 2009 *Nature* **462** 628–632
- [2] Aidelsburger M, Atala M, Nascimbène S, Trotzky S, Chen Y A and Bloch I 2011 *Phys. Rev. Lett.* **107**(25) 255301
- [3] Sørensen A S, Demler E and Lukin M D 2005 *Phys. Rev. Lett.* **94**(8) 086803
- [4] Kolovsky A R 2013 *Journal of Physics B: Atomic, Molecular and Optical Physics* **46** 145301
- [5] Maksimov D N, Chesnokov I Y, Makarov D V and Kolovsky A R 2013 *Journal of Physics B: Atomic, Molecular and Optical Physics* **46** 145302
- [6] Lin Y J, Jimenez-Garcia K and Spielman I B 2011 *Nature* **471** 83–86
- [7] Hamner C, Zhang Y, Khamehchi M A, Davis M J and Engels P 2015 *Phys. Rev. Lett.* **114**(7) 070401
- [8] Sato M, Takahashi Y and Fujimoto S 2009 *Phys. Rev. Lett.* **103**(2) 020401
- [9] Pichler H, Schachenmayer J, Simon J, Zoller P and Daley A J 2012 *Phys. Rev. A* **86**(5) 051605
- [10] Pichler H, Schachenmayer J, Daley A J and Zoller P 2013 *Phys. Rev. A* **87**(3) 033606
- [11] Argonov V Y and Prants S V 2008 *Phys. Rev. A* **78**(4) 043413
- [12] Argonov V Y and Prants S V 2008 *Europhysics Letters* **81** 24003
- [13] Schachenmayer J, Pollet L, Troyer M and Daley A J 2014 *Phys. Rev. A* **89**(1) 011601
- [14] Schrödinger E 1930 *Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl.* **24** 418–428
- [15] Qu C, Hamner C, Gong M, Zhang C and Engels P 2013 *Phys. Rev. A* **88**(2) 021604
- [16] LeBlanc L J, Beeler M C, Jimenez-Garcia K, Perry A R, Sugawa S, Williams R A and Spielman I B 2013 *New Journal of Physics* **15** 073011
- [17] Neiman A and Schimansky-Geier L 1994 *Phys. Rev. Lett.* **72**(19) 2988–2991
- [18] Anishchenko V S, Neiman A B, Moss F and Schimansky-Geier L 1999 *Physics-Uspekhi* **42** 7–36
- [19] Martone G I, Li Y, Pitaevskii L P and Stringari S 2012 *Phys. Rev. A* **86**(6) 063621
- [20] Makarov D V and Kon'kov L E 2013 *Physics Letters A* **377** 3093–3097
- [21] Makarov D V and Konkov L E 2014 *Eur. Phys. J. B* **87** 281
- [22] Makarov D V and Konkov L E 2015 *Physica Scripta* **90** 035204
- [23] Raghavan S, Smerzi A, Fantoni S and Shenoy S R 1999 *Phys. Rev. A* **59**(1) 620–633
- [24] Zibold T, Nicklas E, Gross C and Oberthaler M K 2010 *Phys. Rev. Lett.* **105**(20) 204101
- [25] Uleysky M and Makarov D 2014 *J. Russ. Las. Res.* **35** 138–150 ISSN 1071-2836
- [26] Prants S 1996 *Optics Communications* **125** 222 – 225 ISSN 0030-4018
- [27] Prants S 1997 *Optics and Spectroscopy* **83** 23 – 27
- [28] Prants S V 2009 *Journal of Experimental and Theoretical Physics* **109** 751–761
- [29] Konkov L E and Prants S V 2009 *Journal of Russian Laser Research* **30** 404–410
- [30] D'Errico C, Moratti M, Lucioni E, Tanzi L, Deissler B, Inguscio M, Modugno G, Plenio M B and Caruso F 2013 *New Journal of Physics* **15** 045007
- [31] de Falco D and Tamascelli D 2013 *Journal of Physics A: Mathematical and Theoretical* **46** 225301